جلسه سوم درس معادلات

معادلات ← معادلات خطی

برنولی

کامل

تفکیک پذیر - جدا شدنی

معادلات کامل: اگر معادله دیفرانسیل مرتبه اول به صورت $M(x,y)\,dx + N(x,y)\,dy = 0$ داشته باشیم

و شرط روبرو را داشته باشد $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

به معادله کامل گوییم.

$$u(x,y) = c \longrightarrow du = 0 \longrightarrow \begin{cases} \frac{\partial u}{\partial x} = M \\ \frac{\partial u}{\partial y} = N \end{cases}$$

مثال: معادله دیفرانسیل $\underbrace{(2xy+3)}_{M}\,dx + \underbrace{(x^2+8y)}_{N}\,dy = 0$ را حل کنید.

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2x \\ \frac{\partial N}{\partial x} &= 2x \end{aligned}\right\} \longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow \text{معادله کامل} \longrightarrow \begin{cases} \frac{\partial u}{\partial x} = 2xy+3 \longrightarrow u = \int (2xy+3)\,dx \\ \frac{\partial u}{\partial y} = x^2+8y \quad (1) \end{cases}$$

$$\longrightarrow \boxed{u = \cancel{2}y\frac{x^2}{\cancel{2}} + 3x + f(y)} \; (3) \longrightarrow \frac{\partial u}{\partial y} = x^2 + 0 + f'(y) \; (2) \longrightarrow \cancel{x^2} + 8y = \cancel{x^2} + f'(y)$$

$$\longrightarrow f'(y) = 8y \longrightarrow f(y) = \int 8y\,dy = \frac{8y^2}{2} + c \xrightarrow{(3)} \boxed{u = yx^2 + 3x + 4y^2 + c = 0}$$

مثال: معادله دیفرانسیل $y' = \frac{x^3-y}{x}$ را حل کنید.

$$\frac{dy}{dx} = \frac{x^3-y}{x} \longrightarrow x\,dy = (x^3-y)\,dx \longrightarrow \underbrace{(x^3-y)}_{M}\,dx \underbrace{-x}_{N}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -1 \\ \frac{\partial N}{\partial x} &= -1 \end{aligned}\right\} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow \text{معادله کامل است}$$

$$\underbrace{y' = x^2 - \frac{y}{x}}_{f(x,y)} \longrightarrow y' + \frac{y}{x} = x^2 \longrightarrow p(x) = \frac{1}{x} \quad q(x) = x^2 \longrightarrow$$

از قسم خطی هم حل می‌شود

$$f(\lambda x, \lambda y) = (\lambda x)^2 - \frac{\cancel{\lambda} y}{\cancel{\lambda} x} = \lambda^2 x^2 - \frac{y}{x} \longrightarrow \text{همگن نیست}$$

این معادله تفکیک پذیر هم نیست

$$\begin{cases} \frac{\partial u}{\partial x} = x^3 - y \quad (2) \\ \frac{\partial u}{\partial y} = -x \longrightarrow \boxed{u = \int -x\,dy = -xy + f(x)} \; (3) \longrightarrow \frac{\partial u}{\partial x} = -y + f'(x) \quad (1) \end{cases}$$

$$(1) = (2) \longrightarrow x^3 - \cancel{y} = -\cancel{y} + f'(x) \longrightarrow f'(x) = x^3 \longrightarrow f(x) = \int x^3\,dx = \frac{x^4}{4} + c$$

$$\boxed{-xy + \frac{x^4}{4} + c = 0}$$

مثال: معادله دیفرانسیل $\underbrace{(x^2 - x + y^2)}_{M}dx \underbrace{- (e^y - 2xy)}_{N}dy = 0$ را حل کنید.

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2y\\ \frac{\partial N}{\partial x} &= 2y\end{aligned}\right\} \longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$$ معادله کامل است

$$\frac{\partial u}{\partial x} = x^2 - x + y^2 \longrightarrow u = \int (x^2 - x + y^2)\,dx = \frac{x^3}{3} - \frac{x^2}{2} + y^2x + f(y) \quad (3)$$

$$\frac{\partial u}{\partial y} = 2yx + f'(y) \quad (1)$$

$$\frac{\partial u}{\partial y} = -e^y + 2xy \quad (2)$$

$$\xrightarrow{(1)=(2)} -e^y + \cancel{2xy} = \cancel{2yx} + f'(y) \longrightarrow f'(y) = -e^y \longrightarrow f(y) = -\int e^y dy = -e^y + c$$

$$\xrightarrow{(3)} \boxed{u = \frac{x^3}{3} - \frac{x^2}{2} + y^2x - e^y + c = 0}$$

مثال: معادله دیفرانسیل $(4x^3\sin^3 y - 2x\sin y)\,dx + (3x^4\sin^2 y - x^2)\underbrace{\cos y\,dy}= 0$ را حل کنید.

$$\sin y = P \longrightarrow \cos y\,dy = dP$$

$$\underbrace{(4x^3P^3 - 2xP)}_{M}dx + \underbrace{(3x^4P^2 - x^2)}_{N}dP = 0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial P} &= 12x^3P^2 - 2x\\ \frac{\partial N}{\partial x} &= 12x^3P^2 - 2x\end{aligned}\right\} \frac{\partial M}{\partial P} = \frac{\partial N}{\partial x} \longrightarrow$$ معادله کامل است

$$\frac{\partial u}{\partial x} = 4x^3P^3 - 2xP \longrightarrow u \overset{(3)}{=} \frac{4x^4}{4}P^3 - \frac{2x^2}{2}P + f(P) \longrightarrow \frac{\partial u}{\partial P} = 3x^4P^2 - x^2 + f'(P) \quad (1)$$

$$\frac{\partial u}{\partial P} = 3x^4P^2 - x^2 \quad (2)$$

$$\xrightarrow{(1),(2)} f'(P) = 0 \longrightarrow f(P) = c$$

$$\xrightarrow{(3)} u = x^4P^3 - x^2P + c = 0 \longrightarrow \boxed{x^4\sin^3 y - x^2\sin y + c = 0}$$

فاکتور انتگرال گیری

$$Mdx + Ndy = 0 \quad , \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \overset{?}{F}(Mdx + Ndy) = 0$$

$$FMdx + FNdy = 0 \longrightarrow \frac{\partial (FM)}{\partial y} \overset{(*)}{=} \frac{\partial (FN)}{\partial x}$$

F فاکتوری است که اگر در طرفین معادله اصلی ضرب شود معادله اصلی کامل می‌شود و اثری روی حل‌ها ندارد. ها حل می‌کنیم

$$\frac{\partial F}{\partial y}M + \underbrace{\frac{\partial M}{\partial y}F} = \frac{\partial F}{\partial x}N + \underbrace{\frac{\partial N}{\partial x}F} \longrightarrow F\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\partial F}{\partial x}N - \frac{\partial F}{\partial y}M$$

$$\frac{\partial (FM)}{\partial y} = \frac{\partial (FN)}{\partial x} \longrightarrow F\frac{\partial M}{\partial y} + M\frac{\partial F}{\partial y} = F\frac{\partial N}{\partial x} + N\frac{\partial F}{\partial x}$$

$$\longrightarrow F\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial F}{\partial x} - M\frac{\partial F}{\partial y} \longrightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\frac{\partial F}{\partial x}}{F} - M\frac{\frac{\partial F}{\partial y}}{F}$$

$$\longrightarrow \boxed{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\partial \ln F}{\partial x} - M\frac{\partial \ln F}{\partial y}} \quad (*)$$

اولین مدل فاکتور انتگرال ساز F(x)

در این صورت $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) =$ تابعی از x باشد فاکتور انتگرال ساز $F(x) = e^{\int f(x)dx}$ است

$$\frac{\partial \ln F}{\partial x} = f(x) \longrightarrow \partial \ln F = f(x)\,\partial x \longrightarrow \ln F = \int f(x)\,dx$$

$$\xrightarrow{e^{\ln F} = F} F = e^{\int f(x)dx}$$

مثال: معادله دیفرانسیل $\underbrace{(x+y^2)}_{M}dx \underbrace{-2xy}_{N}dy = 0$ را حل کنید.

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2y \\ \frac{\partial N}{\partial x} &= -2y \end{aligned}\right\} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ کامل نیست

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2xy}\left(2y - (-2y)\right) = \frac{4y}{-2xy} = \frac{-2}{x} =$$ تابعی از x

$$\longrightarrow F = e^{\int \frac{-2}{x}dx} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

M جدید $= M$ تبدیل $F = (x+y^2) \times \frac{1}{x^2} = \frac{1}{x} + \frac{y^2}{x^2}$

N جدید $= N$ تبدیل $F = (-2xy) \times \frac{1}{x^2} = \frac{-2y}{x}$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{2y}{x^2} \\ \frac{\partial N}{\partial x} &= \frac{2y}{x^2} \end{aligned}\right)$$ معادله کامل است

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{1}{x} + \frac{y^2}{x^2} \quad (1) \\ \frac{\partial u}{\partial y} = \frac{-2y}{x} \longrightarrow u = \int \frac{-2y}{x}dy = \frac{-\cancel{2}}{x}\frac{y^2}{\cancel{2}} + f(x) \quad (3) \end{cases}$$

$$\longrightarrow \frac{\partial u}{\partial x} = \frac{y^2}{x^2} + f'(x) \quad (2) \xrightarrow{(1)=(2)} f'(x) = \frac{1}{x} \to f(x) = \int \frac{dx}{x} = \ln x + C$$

$$\longrightarrow \boxed{u = \frac{-y^2}{x} + \ln x + c = 0}$$

جواب کلی

دومین حالت فاکتور انتگرال ساز

$$\frac{1}{-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(y)$$ تابع y باشد

$$F(y) = e^{\int f(y)\,dy}$$

مثال: معادله دیفرانسیل $y(1+xy)\,dx - x\,dy = 0$ را حل کنید.

$$M = y(1+xy) = y + xy^2 \longrightarrow \frac{\partial M}{\partial y} = 1 + 2xy$$

$$N = -x \longrightarrow \frac{\partial N}{\partial x} = -1$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ کامل نیست

$$\frac{1}{-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-y(1+xy)}(1+2xy-(-1)) = \frac{1}{-y(1+xy)}(2+2xy) = \frac{2\cancel{(1+xy)}}{-y\cancel{(1+xy)}}$$

$$= \frac{-2}{y} = \text{تابع } y \text{ است} \longrightarrow F = e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \frac{1}{y^2}$$

$$M \text{ جدید} = y(1+xy) \times \frac{1}{y^2} = \frac{1+xy}{y}$$

$$N \text{ جدید} = -x \times \frac{1}{y^2} = \frac{-x}{y^2}$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{1+xy}{y} = \frac{1}{y} + x \longrightarrow u = \frac{x}{y} + \frac{x^2}{2} + f(y) \quad 3 \\ \frac{\partial u}{\partial y} = -\frac{x}{y^2} \quad 2 \end{cases}$$

$$\longrightarrow \frac{\partial u}{\partial y} = \frac{-x}{y^2} + f'(y) \quad 1$$

$$\xrightarrow{1 = 2} f'(y) = 0 \longrightarrow f(y) = C$$

$$\xrightarrow{3} \boxed{u = \frac{x}{y} + \frac{x^2}{2} + C = 0}$$

سومین حالت فاکتور انتگرال ساز